

ON THE INTEGRABLE GENERALIZATION OF THE 1D TODA LATTICE

P.Yu.Tsyba, K.R.Esmakhanova, G.N.Nugmanova, R.Myrzakulov*
*Eurasian International Center for Theoretical Physics,
 Eurasian National University, Astana, 010008, Kazakhstan*

Abstract

A generalized Toda Lattice equation is considered. The associated linear problem (Lax representation) is found. For simple case $N = 3$ the τ -function Hirota form is presented that allows to construct an exact solutions of the equations of the 1DGTL. The corresponding hierarchy and its relations with the nonlinear Schrodinger equation and Hersteinberg ferromagnetic equation are discussed.

1 Introduction

The 1D Toda lattice (1DTL)

$$\ddot{q}_n = e^{q_{n-1}-q_n} - e^{q_n-q_{n+1}} \quad (1.1)$$

is one of most important integrable equations which plays an important role in mathematics and physics. In physics, the equations (1.1) describe an interacting N particles, each with mass $m_n = 1$, arranged along a line at positions q_1, q_2, \dots, q_N . Between each pair of adjacent particles, there is a force whose magnitude depends exponentially on the distance between them. The 1DTL was discovered by Morikasu Toda in 1967 [1]. Using the computer experiments, in [7] was suggested that the 1DTL is integrable. In [4], [8], [9] the integrability of the 1DTL is proved. Note that this equation is a discrete approximation of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \quad (1.2)$$

The aim of this Letter is to construct the some integrable generalization of the equation (1.1), that can be linked to both linear and quadratic compatible Poisson-brackets for the usual 1DTL.

*The corresponding author. Email: cnlpmmyra1954@yahoo.com

2 Background on 1DTL

In this section we present some known fundamental informations for the 1DTL and fix some notations. Let p_n denotes the momentum of the n th particle. Then the total energy of the system is the Hamiltonian

$$H = \frac{1}{2} \sum_{n=1}^N p_n^2 + \sum_{n=1}^{N-1} e^{q_n - q_{n+1}}. \quad (2.1)$$

So the system (1.1) can be written as

$$\begin{aligned} \dot{q}_n &= \{H, q_n\} = \frac{\partial H}{\partial p_n}, \\ \dot{p}_n &= \{H, p_n\} = -\frac{\partial H}{\partial q_n}. \end{aligned} \quad (2.2)$$

Except the original form (1.1), there are exist its the other various equivalent forms. Some of them as follows.

i)

$$\begin{aligned} \dot{\alpha}_n &= \alpha_n(\beta_n - \beta_{n+1}), \\ \dot{\beta}_n &= \alpha_{n-1} - \alpha_n. \end{aligned} \quad (2.3)$$

ii)

$$\begin{aligned} \dot{a}_n &= a_n(b_{n+1} - b_n), \\ \dot{b}_n &= 2(\alpha_n^2 - \alpha_{n-1}^2). \end{aligned} \quad (2.4)$$

iii)

$$[D_t^2 - 4 \sinh^2(\frac{D_n}{2})]f_n \circ f_n = 0. \quad (2.5)$$

iv)

$$\ddot{\tau}_n \tau_n - \dot{\tau}_n^2 = \tau_{n+1} \tau_{n-1} \quad (2.6)$$

and so on. Above D_t , D_n are the well known Hirota bilinear operators and τ_n is so-called τ -function which play a key role of the theory of integrable systems. Note that a new and initial "physical" (q_n, p_n) dependent variables are related as

$$\begin{aligned} \alpha_n &= e^{q_n - q_{n+1}}, & \beta_n &= p_n, \\ a_n &= \frac{1}{2} e^{\frac{1}{2}(q_n - q_{n+1})}, & b_n &= -\frac{1}{2} p_n, \\ \ln f_n &= (\ln f_n)_{tt} = e^{q_{n-1} - q_n}, \\ q_n &= \ln \frac{\tau_{n-1}}{\tau_n}. \end{aligned} \quad (2.7)$$

There are at least two possible Lax representations for the 1DTL, one of order 2×2 , another one of order $N \times N$ (see, e.i. [10]). The 2×2 Lax pair is defined as

$$L_n = \begin{pmatrix} p_n + \lambda & e^{q_n} \\ -e^{-q_n} & 0 \end{pmatrix}, \quad M_n = \begin{pmatrix} 0 & -e^{q_n} \\ e^{-q_n} & \lambda \end{pmatrix}. \quad (2.8)$$

The associated linear problem is

$$\begin{aligned} \Psi_{n+1} &= L_n \Psi_n, \\ \Psi_{nt} &= M_n \Psi_n. \end{aligned} \quad (2.9)$$

The compatibility condition of these equations

$$L_{nt} + L_n M_n - M_{n+1} L_n = 0 \quad (2.10)$$

gives the equation (1.1). The equivalent Lax representation is given by the $N \times N$ Lax pair

$$L_n = \sum_{\nu=1}^N L^\nu X_\nu, \quad M_n = \sum_{\nu=1}^N M^\nu X_\nu, \quad (2.11)$$

where $[X_\mu, X_\nu] = C_{\mu\nu}^\lambda X_\lambda$. Then the TL equation is obtained from the Lax equation

$$\dot{L} = [L, M]. \quad (2.12)$$

There are exist a so-called r-matrix representation for the Poisson brackets $\{L^\mu, L^\nu\}$ between the matrix elements of L . It has the form

$$\{L \otimes, L\} = [r, L \otimes I + I \otimes L] = [r, L_1 + L_2] \quad (2.13)$$

or

$$\{L \otimes, L\} = [r, L \otimes I] - [r^T, I \otimes L] = [r, L_1 + L_2], \quad (2.14)$$

where

$$r = \sum_{\mu, \nu=1}^N r^{\mu\nu} X_\mu \otimes X_\nu, \quad r^T = \sum_{\mu, \nu=1}^N r^{\nu\mu} X_\mu \otimes X_\nu, \quad L_1 = L \otimes I, \quad L_2 = I \otimes L. \quad (2.15)$$

Finally we have

$$\{L \otimes, L\} = (r^{\tau\nu} C_{\tau\lambda}^\mu L^\lambda - r^{\tau\mu} C_{\tau\lambda}^\nu L^\lambda) X_\mu \otimes X_\nu. \quad (2.16)$$

In this notes we use the following form of the r-matrix

$$r = \sum_{i=1}^N E_{ii} \otimes E_{ii} + 2 \sum_{i,j=1(i < j)}^N E_{ij} \otimes E_{ji}. \quad (2.17)$$

3 Generalized TL

In this paper we deal with the system of nonlinear differential-difference equations

$$\begin{aligned} \dot{p}_k &= 2(a_{k-1}b_{k-1} - a_k b_k) + 2uv(\delta_{k1} - \delta_{k,-1}), \\ \dot{a}_k &= (p_k - p_{k+1})a_k + 2v(b_{k-1}\delta_{k,0} - b_{k+1}\delta_{k,-1}), \\ \dot{b}_k &= (p_k - p_{k+1})b_k + 2u(a_{k-1}\delta_{k,0} - a_{k+1}\delta_{k,-1}), \\ \dot{u} &= (p_2 - p_4)u, \\ \dot{v} &= (p_2 - p_4)v. \end{aligned} \quad (3.1)$$

It transforms to the ordinary TL as $u = v = 0$ since we call it as the generalized TL (1DGTL). The system (3.1) is integrable as it can be written in Lax form as (2.11)

with

$$L = \begin{pmatrix} p_{-N} & a_{-N} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ b_{-N} & \ddots & \ddots & \ddots & & & & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & & & & \vdots \\ \vdots & \ddots & \ddots & p_{-1} & a_{-1} & v & & & \vdots \\ \vdots & & 0 & b_{-1} & p_0 & a_0 & 0 & & \vdots \\ \vdots & & & u & b_0 & p_1 & \ddots & \ddots & \vdots \\ \vdots & & & & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & \ddots & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & b_{N-1} & p_N \end{pmatrix}, \quad (3.2)$$

$$M = \begin{pmatrix} 0 & a_{-N} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ -b_{-N} & \ddots & \ddots & \ddots & & & & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & & & & \vdots \\ \vdots & \ddots & \ddots & 0 & a_{-1} & v & & & \vdots \\ \vdots & & 0 & -b_{-1} & 0 & a_0 & 0 & & \vdots \\ \vdots & & & -u & -b_0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & & & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & \ddots & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -b_{N-1} & 0 \end{pmatrix}. \quad (3.3)$$

There are take place a Lie-Poisson brackets (2.13) between elements of L which are given by

$$\begin{aligned} \{p_i, a_i\} &= a_i, \\ \{p_{i+1}, a_i\} &= -a_i, \\ \{p_i, b_i\} &= b_i, \\ \{p_{i+1}, b_i\} &= -b_i, \\ \{p_{-1}, u\}_M &= u, \\ \{p_1, u\}_M &= -u, \\ \{p_{-1}, v\}_M &= v, \\ \{p_1, v\}_M &= -v, \\ \{a_{-1}, a_0\}_M &= 2v, \\ \{b_{-1}, b_0\}_M &= 2u. \end{aligned} \quad (3.4)$$

All other brackets are zero. We denote this bracket by π_1 . The functions

$$H_i = \frac{1}{i} \text{tr} L^i \quad (3.5)$$

are independent invariants in involution that is

$$\{H_i, H_j\} = 0. \quad (3.6)$$

The expressions for H_i are, for example,

$$\begin{aligned} H_1 &= \sum_{k=-N}^N p_k, \\ H_2 &= \frac{1}{2} \sum_{k=-N}^N p_k^2 + \sum_{k=-(N-1)}^{N-1} a_k b_k + uv, \\ H_3 &= \frac{1}{3} \sum_{k=-N}^N p_k^3 + \sum_{k=1}^N [a_1 b_1 p_1 + (a_1 b_1 + a_2 b_2 + uv) p_2 + (a_2 b_2 + a_3 b_3) p_3 + \\ &\quad (a_3 b_3 + a_4 b_4 + uv) p_4 + a_4 b_4 p_5 + a_2 a_3 u + b_2 b_3 v], \end{aligned} \tag{3.7}$$

and so on. The invariant H_1 is the only Casimir. The Hamiltonian in this bracket is $H_2 = \frac{1}{2} \text{tr } L^2$.

As for the usual TL in our case we can introduce the quadratic Toda brackets which appears in conjunction with isospectral deformations of Jacobi matrices. It is a Poisson bracket in which the Hamiltonian vector field generated by H_1 is the same as the Hamiltonian vector field generated by H_2 with respect to the π_1 bracket. We will denote this Poisson bracket by π_2 . The bracket π_2 is easily defined by taking the Lie derivative of π_1 in the direction of suitable master symmetry. This bracket has $\det L$ as Casimir and $H_1 = \text{tr } L$ is the Hamiltonian. The eigenvalues of L are still in involution. Furthermore, π_2 is compatible with π_1 . We also have

$$\pi_2 dH_i = \pi_1 dH_{i+1}. \tag{3.8}$$

Note that both brackets π_1 and π_2 transforms to the corresponding brackets of the usual TL for the case $u = v = 0$.

4 Case N=3

Let us now we consider in more detail the case $N = 3$. For this case the corresponding equations of the 1DGTL take the forms

$$\begin{aligned} \dot{p}_1 &= -2(a_1^2 + u^2), \\ \dot{p}_2 &= 2(a_1^2 - a_2^2), \\ \dot{p}_3 &= 2(a_2^2 + u^2), \\ \dot{a}_1 &= a_1(p_1 - p_2) - 2ua_2, \\ \dot{a}_2 &= a_2(p_2 - p_3) + 2ua_1, \\ \dot{u} &= (p_1 - p_3)u. \end{aligned} \tag{4.1}$$

This system can be written in Lax form as

$$\dot{L} = [L, M], \tag{4.2}$$

where

$$L = \begin{pmatrix} p_1 & a_1 & u \\ a_1 & p_2 & a_2 \\ u & a_2 & p_3 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & a_1 & u \\ -a_1 & 0 & a_2 \\ -u & -a_2 & 0 \end{pmatrix}. \tag{4.3}$$

There exists a Lie-Poisson bracket given by the formula

$$\begin{aligned} \{p_i, a_i\} &= a_i, \\ \{p_{i+1}, a_i\} &= -a_i, \\ \{p_1, u\} &= u, \\ \{p_3, u\} &= -u, \\ \{a_1, a_2\} &= 2u. \end{aligned} \tag{4.4}$$

All other brackets are zero. We denote this bracket by π_1 . The functions

$$H_i = \frac{1}{i} \text{tr} L^i \quad (4.5)$$

are independent invariants in involution that is

$$\{H_i, H_j\} = 0.$$

The expressions for H_i are, for example,

$$\begin{aligned} H_1 &= \sum_{k=1}^3 p_k, \\ H_2 &= \frac{1}{2} \sum_{k=1}^3 p_k^2 + \sum_{k=1}^2 a_k^2 + u^2, \\ H_3 &= \frac{1}{3} \sum_{k=1}^3 p_k^3 + \sum_{k=1}^3 [a_1^2 p_1 + (a_1^2 + a_2^2 + u^2) p_2 + (a_2^2 + a_3^2) p_3 + \\ &\quad (a_3^2 + a_4^2 + u^2) p_4 + a_4^2 p_5 + 2a_2 a_3 u] \end{aligned} \quad (4.6)$$

and so on. The Hamiltonian in this bracket is $H_2 = \frac{1}{2} \text{tr} L^2$. The Casimirs of the system are

$$\begin{aligned} C_1 &= H_1 = p_1 + p_2 + p_3, \\ C_2 &= \frac{a_1 a_2}{u} - 2p_2, \\ C_3 &= \frac{v}{u} = d_3. \end{aligned} \quad (4.7)$$

Note that

$$\dot{C}_k = 0.$$

4.1 The q_k, p_k coordinates

Let us rewrite the system (4.1) in terms of the coordinates q_k , $p_k = \dot{q}_k$, $k = 1, 2, 3$. Then from (4.1) we have

$$\begin{aligned} \ddot{q}_1 &= -2(a_1^2 + u^2), \\ \ddot{q}_2 &= -2(a_2^2 - a_1^2), \\ \ddot{q}_3 &= 2(a_2^2 + u^2), \\ \dot{a}_1 &= a_1(p_1 - p_2) - 2ua_2, \\ \dot{a}_2 &= a_2(p_2 - p_3) + 2ua_1, \\ \dot{u} &= p_{13}u. \end{aligned} \quad (4.8)$$

Hence we get

$$u = u_0 + e^{q_{13}}, \quad q_{ij} = q_i - q_j, \quad p_{ij} = p_i - p_j, \quad u_0 = \text{const}(t). \quad (4.9)$$

Let

$$a_1 = e^{q_{12}} p_4, \quad a_2 = e^{q_{23}} q_4.$$

Then

$$L = \begin{pmatrix} p_1 & e^{q_{12}} p_4 & e^{q_{13}} \\ d_1 e^{q_{12}} p_5 & p_2 & e^{q_{23}} q_4 \\ d_1 d_2 e^{q_{13}} & d_2 e^{q_{23}} q_5 & p_3 \end{pmatrix}, \quad (4.10)$$

and the equations of motion become

$$\begin{aligned}
\ddot{q}_1 &= -2(a_1^2 + u^2), \\
\ddot{q}_2 &= -2(a_2^2 - a_1^2), \\
\ddot{q}_3 &= 2(a_2^2 + u^2), \\
\dot{p}_4 &= -2e^{q_{13}+q_{23}-q_{12}}q_4 = -2e^{2q_{23}}q_4, \\
\dot{q}_4 &= 2e^{q_{13}-q_{23}+q_{12}}p_4 = 2e^{2q_{12}}p_4, \\
\dot{u} &= p_{13}u.
\end{aligned} \tag{4.11}$$

So we see that $p_4 = \frac{1}{2}e^{2q_{21}}\dot{q}_4 \neq \dot{q}_4$. Finally we have

$$\begin{aligned}
\ddot{q}_1 &= -2(e^{2q_{12}}p_4^2 + e^{2q_{13}}), \\
\ddot{q}_2 &= -2(e^{2q_{23}}q_4^2 - e^{2q_{12}}p_4^2), \\
\ddot{q}_3 &= 2(e^{2q_{23}}q_4^2 + e^{2q_{13}}), \\
\ddot{q}_4 &= 4(p_{12}p_4e^{2q_{12}} - q_4e^{2q_{13}}), \\
\dot{u} &= p_{13}u
\end{aligned} \tag{4.12}$$

or

$$\begin{aligned}
\ddot{q}_1 &= -2(p_4^2e^{2q_{12}} + e^{2q_{13}}), \\
\ddot{q}_2 &= -2(q_4^2e^{2q_{23}} - p_4^2e^{2q_{12}}), \\
\ddot{q}_4 &= 4(p_{12}p_4e^{2q_{12}} - q_4e^{2q_{13}}).
\end{aligned} \tag{4.13}$$

Here

$$C_4 = p_4q_4 + \alpha e^{p_4q_4}. \tag{4.14}$$

4.2 The (P_k, Q_k) coordinates

Let us consider a new representation for the Lax matrix L as

$$L = \begin{pmatrix} P_1 & e^{Q_1}P_3 & e^{Q_1+Q_2} \\ d_1e^{Q_1}P_4 & P_2 - P_1 & e^{Q_2}Q_3 \\ d_1d_2e^{Q_1+Q_2} & d_2e^{Q_2}Q_4 & -P_2 \end{pmatrix}, \tag{4.15}$$

where $\{P_i, Q_j\} = \delta_{ij}$. In this case the equations of motion takes the form

$$\begin{aligned}
\dot{P}_1 &= Q_1, \\
\dot{P}_2 &= Q_2, \\
\dot{P}_3 &= Q_3, \\
\dot{Q}_1 &= a_1(p_1 - p_2) - 2ua_2, \\
\dot{Q}_2 &= a_2(p_2 - p_3) + 2ua_1, \\
\dot{Q}_2 &= a_2(p_2 - p_3) + 2ua_1, \\
\dot{u} &= (P_1 - P_3)u.
\end{aligned} \tag{4.16}$$

5 Solutions

To find solutions we first introduce a new variables as

$$c_k(t) = p_k\left(\frac{1}{2}t\right), \quad d_{k+1}(t) = a_k^2\left(\frac{1}{2}t\right), \quad w = u^2\left(\frac{1}{2}t\right). \tag{5.1}$$

Then the system (4.1) becomes

$$\begin{aligned}
\dot{c}_1 &= -(d_2 + w), \\
\dot{c}_2 &= d_2 - d_3, \\
\dot{c}_3 &= d_3 + w \\
\dot{d}_2 &= c_{12}d_2 - 2\sqrt{wd_3}, \\
\dot{d}_3 &= c_{23}d_3 + 2\sqrt{wd_2}, \\
\dot{w} &= c_{13}w.
\end{aligned} \tag{5.2}$$

Note that the GTL equation (4.18) is the isospectral ($\lambda_t = 0$) and is the compatibility condition of the following spectral problems

$$\begin{aligned}
\dot{\phi}_n &= c_{n-1}\phi_n + \phi_{n-1}, \\
d_n\phi_{n+1} + c_{n-1}\phi_n + \phi_{n-1} &= \lambda\phi_n,
\end{aligned} \tag{5.3}$$

where λ is a spectral parameter. This means that it has the Lax representation $L_t = [M, L]$, where the Lax pair is defined by

$$\begin{aligned}
L_{nm} &= d_n\delta_{n+1,m} + c_{n-1}\delta_{nm} + \delta_{n-1,m}, \\
M_{nm} &= \delta_{n+1,m} + c_{n-1}\delta_{nm}.
\end{aligned} \tag{5.4}$$

In the matrix form these matrices have the form

$$L = \begin{pmatrix} c_0 & d_1 & w \\ 1 & c_1 & d_2 \\ 0 & 1 & c_2 \end{pmatrix}, \quad M = \begin{pmatrix} c_0 & 1 & 0 \\ 0 & c_1 & 1 \\ 0 & 0 & c_2 \end{pmatrix}, \tag{5.5}$$

If we introduce the following τ -function form of the dependent variable

$$d_k = 1 + (\ln \tau_k)_{tt} = (\ln \tau_k e^{\frac{1}{2}t^2})_{tt}, \quad w = -1 - (\ln f)_{tt} = -(\ln f e^{\frac{1}{2}t^2})_{tt}, \tag{5.6}$$

then we have

$$\begin{aligned}
c_1 &= I_1 - (\ln \frac{\tau_2}{f})_t, \\
c_2 &= I_2 + (\ln \frac{\tau_2}{\tau_3})_t, \\
c_3 &= I_3 + (\ln \frac{\tau_3}{f})_t
\end{aligned} \tag{5.7}$$

and hence

$$\begin{aligned}
c_{12} &= I_{12} - (\ln \frac{\tau_2^2}{\tau_3 f})_t, \\
c_{23} &= I_{23} + (\ln \frac{\tau_2 f}{\tau_3^2})_t, \\
c_{13} &= I_{13} - (\ln \frac{\tau_2 \tau_3}{f^2})_t.
\end{aligned} \tag{5.8}$$

To define the unknown functions we have the system

$$\begin{aligned}
(d_2 + d_3)_t &= c_{12}d_2 + c_{23}d_3, \\
(\dot{d}_3 - c_{23}d_3)^2 &= 4wd_2, \\
\dot{w} - c_{13}w &= 0.
\end{aligned} \tag{5.9}$$

Hence and from (5.6) and (5.8) we get

$$\begin{aligned}
(\ln \tau_2 \tau_3)_{ttt} - [I_{12} - (\ln \frac{\tau_2^2}{\tau_3 f})_t](\ln \tau_2 e^{\frac{1}{2}t^2})_{tt} - [I_{23} + (\ln \frac{\tau_2 f}{\tau_3^2})_t](\ln \tau_3 e^{\frac{1}{2}t^2})_{tt} &= 0, \\
\{(\ln \tau_3)_{ttt} - [I_{23} + (\ln \frac{\tau_2 f}{\tau_3^2})_t](\ln e^{\frac{1}{2}t^2} \tau_3)_{tt}\}^2 + 4(\ln \tau_2 e^{\frac{1}{2}t^2})_{tt}(\ln f e^{\frac{1}{2}t^2})_{tt} &= 0, \\
\ddot{f}f - \dot{f}^2 + f^2 + \frac{f^4}{\tau_2 \tau_3} e^{I_{130} + I_{13t}} &= 0.
\end{aligned} \tag{5.10}$$

For simplicity we set $I_{ij} = 0$. Then the system (5.10) becomes

$$\begin{aligned} (\ln \tau_2 \tau_3)_{ttt} + (\ln \frac{\tau_2^2}{\tau_3 f})_t (\ln \tau_2 e^{\frac{1}{2}t^2})_{tt} - (\ln \frac{\tau_2 f}{\tau_3})_t (\ln \tau_3 e^{\frac{1}{2}t^2})_{tt} &= 0 \\ [(\ln \tau_3)_{ttt} - (\ln \frac{\tau_2 f}{\tau_3})_t (\ln e^{\frac{1}{2}t^2} \tau_3)_{tt}]^2 + 4(\ln \tau_k e^{\frac{1}{2}t^2})_{tt} (\ln f e^{\frac{1}{2}t^2})_{tt} &= 0 \\ \ddot{f}f - \dot{f}^2 + f^2 + \frac{f^4}{\tau_2 \tau_3} e^{I_{130}} &= 0. \end{aligned} \quad (5.11)$$

We expand the functions τ_n, f in a formal power series in an arbitrary parameter ϵ as

$$\tau_n = \sum_{k=0} \epsilon^k \tau_n^{(k)}, \quad f = \sum_{k=0} \epsilon^k f^{(k)}. \quad (5.12)$$

Expanding the l.h.s. of (5.11) in ϵ and equating corresponding coefficients, we get the resulting equations for the functions $\tau_n^{(k)}, f^{(k)}$. Solving these equations we can construct an exact solutions of the underlying set of equations for the 1DGTL.

6 The 1DGTL hierarchy

First let us recall the main formulas of the usual TL hierarchy. The corresponding hierarchy is defined by

$$\frac{\partial L}{\partial t_k} = [L, B_k], \quad B_k = (L^k)_-, \quad k = 1, 2, 3, \dots \quad (6.1)$$

The τ -functions of the TL hierarchy obey the following equations

$$[D_k - h_k(\mathbf{D})]\tau_{n+1} \cdot \tau_n = 0, \quad k = 2, 3, 4, \dots \quad (6.2)$$

Here

$$e^{\sum_{k=1}^{\infty} \frac{1}{k} D_k z^k} = \sum_{n=0}^{\infty} h_n(\mathbf{D}) z^n, \quad \mathbf{D} = (D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \frac{1}{4}D_4, \dots). \quad (6.3)$$

It is interesting to note that the nonlinear Schrodinger equation (NLSE) is the second member of the TL hierarchy. In fact from (6.2) as $k = 2$ and from (2.6) we get the following set of equations (see for example [13])

$$\begin{aligned} (D_2 - D_1^2)\tau_{n+1} \cdot \tau_n &= 0, \\ D_1^2 \tau_n \cdot \tau_n &= 2\tau_{n+1} \cdot \tau_{n-1}. \end{aligned} \quad (6.4)$$

This set is equivalent to the NLSE [13]

$$i\phi_{t_2} + \phi_{t_1 t_1} + 2\phi^2 \bar{\phi} = 0, \quad (6.5)$$

where

$$\phi = \tau_{n+1} \tau_n^{-1}, \quad \bar{\phi} = \tau_{n-1} \tau_n^{-1}, \quad t_2 \longrightarrow it_2. \quad (6.6)$$

The set (6.4) is the compatibility condition of the following set of linear equations

$$\psi_{t_1} = \begin{pmatrix} 0 & \tau_{n-1} \tau_n^{-1} \\ \tau_{n+1} \tau_n^{-1} & 0 \end{pmatrix} \psi, \quad \psi_{t_2} = i \begin{pmatrix} \tau_{n-1} \tau_{n+1} \tau_n^{-2} & -(\tau_{n-1} \tau_n^{-1})_{t_1} \\ (\tau_{n+1} \tau_n^{-1})_{t_1} & -\tau_{n-1} \tau_{n+1} \tau_n^{-2} \end{pmatrix} \psi. \quad (6.7)$$

Note that in this case the matrix $S = \psi^{-1} \sigma_3 \psi$ obeys the Heisenberg ferromagnetic equation

$$2iS_{t_2} = [S, S_{t_1 t_1}]. \quad (6.8)$$

Finally note that the 1DGTL hierarchy has the same form as (6.1) but content the additional two equations.

7 Conclusion

In the present Letter we considered one of integrable generalizations of TL. The corresponding Lax representation is presented. For the particular case $N = 3$ the bilinear form (τ -function form) is found that allows to construct exact solutions of the studied generalized Toda equation.

Acknowledgments

One of the authors (R.M.) thanks N. Reshetikhin for the statement of the problem and for very helpful discussions. R.M. also would like to thank N. Reshetikhin and Department of Mathematics, UC, Berkeley for their hospitality during his visit (January-May, 2009).

References

- [1] Toda M. Vibration of a chain with nonlinear interaction. *J. Phys. Soc. Japan*, **22** (1967), 431-436.
- [2] Reshetikhin N. Characteristic Systems on Poisson Lie Groups and Their Quantization. *CRM Proceedings and Lecture Notes*, **26** (2000), 165-188
- [3] Hoffmann T., Kellendonk J., Kutz N., Reshetikhin N. Factorization Dynamics and Coxeter-Toda Lattices. *Comm. Math. Phys.*, **212** (2000), 297-321
- [4] Flaschka H. The Toda lattice I. Existence of integrals, *Phys. Rev. B* **9** (1974), 1924-1925.
- [5] Flaschka H. On the Toda lattice II. Inverse scattering solution, *Progr. Theor. Phys.* **51** (1974), 703-716.
- [6] Fokas A. S., Fuchssteiner B. The hierarchy of the Benjamin-Ono equation, *Phys. Lett. A* **86** (1981), 341-345.
- [7] Ford J., Stoddard S. D., Turner J. S. On the Integrability of the Toda lattice, *Progr. Theor. Phys.* **50** (1973), 1547-1560.
- [8] Henon M. Integrals of the Toda lattice, *Phys. Rev. B* **9** (1974), 1921-1923.
- [9] Manakov S. V. Complete integrability and stochastization of discrete dynamical systems, *Zh. Exp. Teor. Fiz.* **67** (1974), 543-555.
- [10] Suris Yu.B. The Problem of Integrable Discretization, Hamiltonian Approach. Progress in Mathematics, Vol.219, Birkhäuser, Basel, 2003.
- [11] Weinstein A. The local structure of Poisson manifolds, *J. Diff. Geom.*, **18** (1983), 523-557.

- [12] Strachan I.A.A.B. The dispersive self-dual einstein equations and the Toda Lattice [hep-th/9606101]
- [13] Kodama Y., Pierce V.U. Combinatorics of dispersionless integrable systems and universality in random matrix theory [arXiv:0811.0351]